



## Overview course module “Stochastic Modelling”

I. Introduction

II. Actor-based models for network evolution

III. Co-evolution models for networks and behaviour

**IV. Exponential Random Graph Models**

A. Definition

B. Example: gossip about the boss

C. Estimation / goodness of fit issues



## A: Exponential Random Graph Models

- › history: Frank & Strauss “Markov random graph” model (1986), Frank (1991) and Wasserman & Pattison (1996) generalised to exponential family distribution:

$$\Pr(X = \mathbf{x}) \propto \exp\left(\sum_{k=1}^K \beta_k s_k(\mathbf{x})\right)$$

‘proportional to’  
*proportionality constant unknown  
and practically uncalculable(!)*

linear combination  
of network statistics  
*very flexible!*

*Likelihood of macro structure  $\mathbf{x}$  is explained by prevalence of micro structures  $\mathbf{s}(\mathbf{x})$ , testable via parameters  $\beta$ .*



## Some possible statistics

- tie count statistic



$$s_k(x) = \sum_{i,j=1}^n x_{ij}$$

- reciprocity statistic  
(for directed graphs)



$$s_k(x) = \sum_{i,j=1}^n x_{ij}x_{ji}$$

- transitive closure



$$s_k(x) = \sum_{i,j,k=1}^n x_{ij}x_{jk}x_{ik}$$

A positive parameter  $\beta$  attached to any of these statistics means: the right configuration is more likely than the left one.



## Example

- › Consider an ERG model for an undirected network with 3 parameters for these statistics:

(1) number of edges  $s_{edges}(x) = \frac{1}{2} \sum_{i,j=1}^n x_{ij}$

$s_{2-stars}(x) = \frac{1}{2} \sum_{i,j,k=1}^n x_{ij}x_{ik}$  number of 2-stars (2)

(3) number of triangles  $s_{triangles}(x) = \frac{1}{6} \sum_{i,j,k=1}^n x_{ij}x_{jk}x_{ik}$

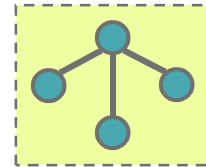
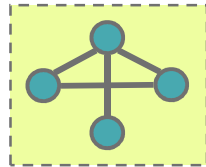
- › Then the 3-parameter ERG distribution function is this one:

$$\begin{aligned} \Pr(X = x) \propto & \exp(\beta_{edges} \times s_{edges}(x) \\ & + \beta_{2-stars} \times s_{2-stars}(x) \\ & + \beta_{triangles} \times s_{triangles}(x)) \end{aligned}$$



## Example

- › ...and consider the following two 4-node-networks & their statistics:



$x^a$

$x^b$

$S_{edges}$

4

3

$S_{2-stars}$

5

3

$S_{triangles}$

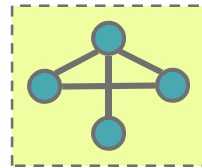
1

0

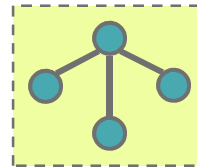


## Example

- Probabilities are (only) given up to a proportionality factor:



$x^a$



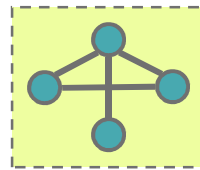
$x^b$

$$\Pr(x^a) \propto \exp(4 \times \beta_{edges} + 5 \times \beta_{2-stars} + 1 \times \beta_{triangles})$$

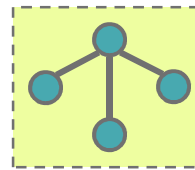
$$\Pr(x^b) \propto \exp(3 \times \beta_{edges} + 3 \times \beta_{2-stars})$$



*...but the ratio of probabilities can be calculated!*



$x^a$



$x^b$

$$\begin{aligned} \frac{\Pr(x^a)}{\Pr(x^b)} &= \frac{\exp(4\beta_{edges} + 5\beta_{2-stars} + \beta_{triangles})}{\exp(3\beta_{edges} + 3\beta_{2-stars})} \\ &= \exp((4-3)\beta_{edges} + (5-3)\beta_{2-stars} + \beta_{triangles}) \\ &= \exp(\beta_{edges} + 2\beta_{2-stars} + \beta_{triangles}) \end{aligned}$$



## Local characterisation of ERGMs

- › ERG distribution is considered as a collection of local conditional tie probabilities

$$\frac{\Pr(x^a)}{\Pr(x^b)} = \exp \left( \sum_{k=1}^K \beta_k \left( s_k(x^a) - s_k(x^b) \right) \right)$$

compared are a network  $x^a$   
and a “neighbouring network”  $x^b$

differences in model  
statistics between  $x^a$  and  $x^b$

*What this equation refers to is  
the (conditional) probability of  
the one tie by which they differ!*

model  
parameters



## Recall similar formula for dynamic networks

- Local characterisation of ERG models:

$$\frac{\Pr(x^a)}{\Pr(x^b)} = \exp\left(\sum_{k=1}^K \beta_k (s_k(x^a) - s_k(x^b))\right)$$

- Local characterisation for actor-based network evolution models:

$$\frac{\Pr(x^c \xrightarrow{i} x^a)}{\Pr(x^c \xrightarrow{i} x^b)} = \exp\left(\sum_{k=1}^K \beta_k (s_{ik}(x^a) - s_{ik}(x^b))\right)$$

compared are two moves ('micro steps')  
made by actor ***i*** from a network ***x<sup>c</sup>*** to two  
"neighbouring networks" ***x<sup>a</sup>*** and ***x<sup>b</sup>***

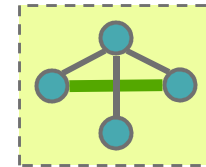
model  
parameters

difference in model statis-  
tics of actor *i* between the  
two compared moves



› ...so how do these ‘conditional odds’ for the **middle tie** to exist (vs. not to exist) look like in applications?

- Suppose, in a (larger) trade network, estimation gave:
  - redundant ties are avoided:  $\beta_{triangles} = -0.4$
  - positive degree variance:  $\beta_{2-stars} = 0.1$
  - low density:  $\beta_{edges} = -1.5$

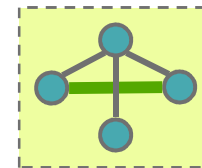




- Then the equation becomes:

$$\begin{aligned}\frac{\text{Pr}(\text{middle tie})}{\text{Pr}(\text{no middle tie})} &= \exp(\beta_{\text{edges}} + 2\beta_{2\text{-stars}} + \beta_{\text{triangles}}) \\ &= \exp(-1.5 + 2 \times 0.1 - 0.4) \\ &= \exp(-1.7) \\ &\approx 0.183 \\ &\approx 1/5.5\end{aligned}$$

i.e., **the middle tie** is about  
5.5 times as likely NOT to exist  
as to exist [given the rest of the network].





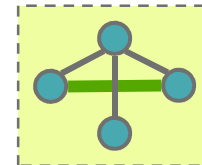
› Same model in a different application:

- Suppose, in a (larger) friendship network, estimates were:

- closure:  $\beta_{triangles} = 0.4$

- pos. degree variance:  $\beta_{2-stars} = 0.1$

- low density:  $\beta_{edges} = -1.5$



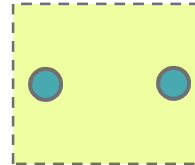
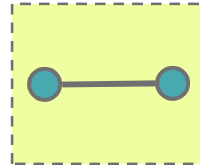
- Then the equation becomes:

$$\frac{\text{Pr}(\text{middle tie})}{\text{Pr}(\text{no middle tie})} = \exp(-0.9) \approx 0.407 \approx 1/2.5$$

- Still two<sup>1/2</sup> times as likely NOT to exist – but always keep in mind that all ties are random...



...random ties are benchmark for evaluating such conditional odds!  
Hence, consider these “networks” as baseline comparison:



$$\frac{\text{Pr}(\text{tie})}{\text{Pr}(\text{no tie})} = \exp(\beta_{\text{edges}}) = \exp(-1.5) \approx 0.223 \approx 1/4.5$$

- › In the *friendship context* (odds=2.5),  
the tie in question is *relatively likely* to be present,
- › In the *trade context* (odds=5.5),  
it is *relatively unlikely* to be present, given the network’s density.



## *Exponential Random Graph Models*

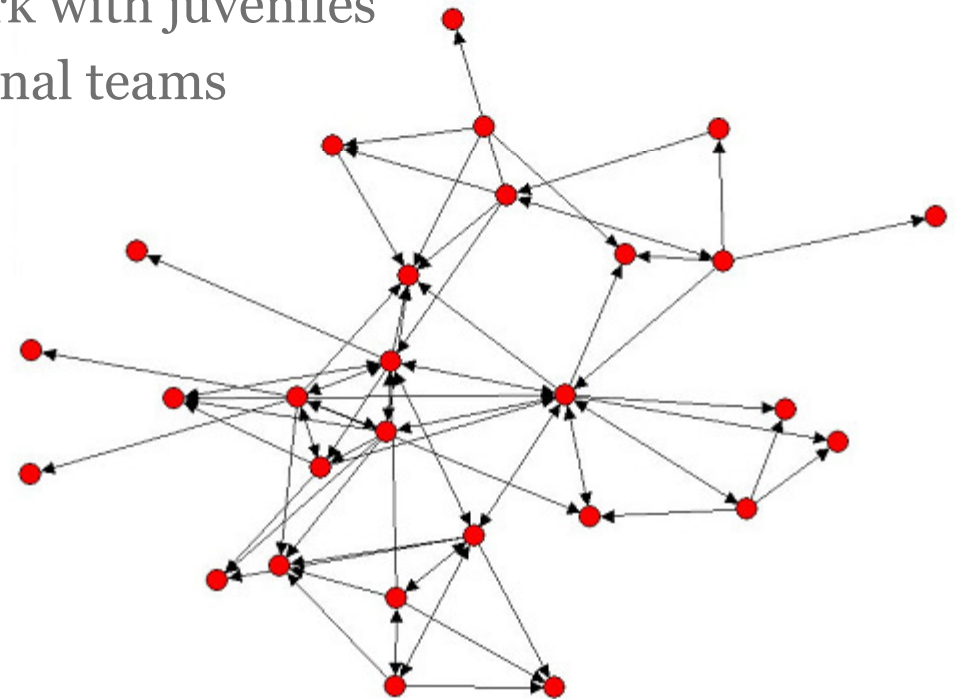
- › high flexibility due to the many possibilities of choosing statistics & controlling effects for each other +
- › can express small worlds (see Robins et al. 2005) but also other network types
- › network probability model: not tied to any particular algorithm +
- › problems: estimation, model specification, interpretation -



## B: Example: gossip about the boss

- › data from Lea Ellwardt (2008)
  - organisation for social work with juveniles
  - one department with internal teams
  - 28 employees

- › *gossip about the boss in this department was largely negative*





## Hypotheses:

1. closeness / proximity is associated with gossip
  - friendship / liking predicts gossip
  - communication frequency predicts gossip
  - team structure predicts gossip
  - gossip occurs in local clusters
2. information asymmetry is associated with gossip
  - dissimilarity in / lack of contacts with the boss predicts g'sp.
3. “negative attitude” is associated with gossip
  - distrust in management predicts gossip
  - disliking the boss predicts gossip
4. attitude similarity is associated with gossip



## Results from an exponential random graph analysis:

<i>Parameter</i>	<i>Estimate</i>	<i>SE</i>		
<b>NETWORK EFFECTS</b>				
1. Reciprocity	0.9356	0.6254		
2. Transitive triplets	-0.1851	0.2873	... (1)	
3. 3-cycles	-0.1362	0.3619	... (1)	
4. Alternating out-k-stars	<b>0.8823*</b>	<b>0.3179</b>	← - - -	<b>Unpredicted: heterogeneity of outdegrees</b>
5. Alternating in-k-stars	-0.2607	0.3701		
6. Alternating k-triangles	<b>0.9762*</b>	<b>0.4834</b>	✓ (1)	
7. Team membership	<b>0.7866*</b>	<b>0.2835</b>	✓ (1)	
8. Communication ego-alter (symmetric)	<b>0.9111*</b>	<b>0.3254</b>	✓ (1)	
9. Liking of alter by ego (out-degree)	<b>1.8782*</b>	<b>0.3305</b>	✓ (1)	



## *Results from an exponential random graph analysis:*

<i>Parameter</i>	<i>Estimate</i>	<i>SE</i>	
SENDER EFFECTS			
10. Trust in management	-0.1261	0.1337	... (3)
11. Liking of manager	<b>-0.5071*</b>	<b>0.2289</b>	✓ (3)
12. Communication with manager	<b>-0.3239*</b>	<b>0.0915</b>	✓ (2)
RECEIVER EFFECTS			
13. Trust in management	0.2187	0.1996	... (3)
14. Liking of manager	0.2244	0.3588	... (3)
15. Communication with manager	-0.0445	0.1411	... (2)



## *Results from an exponential random graph analysis:*

<i>Parameter</i>	<i>Estimate</i>	<i>SE</i>	
SIMILARITY EFFECTS			
16. Trust in management (similar)	<b>-1.3708*</b>	<b>0.7921</b>	<b>✓ (4)</b>
17. Liking of manager (same)	0.3919	0.2878	<b>... (4)</b>
18. Communication with manager (similar)	-0.4359	0.6343	<b>... (4)</b>

\*  $p < 0.05$  (one-sided test)

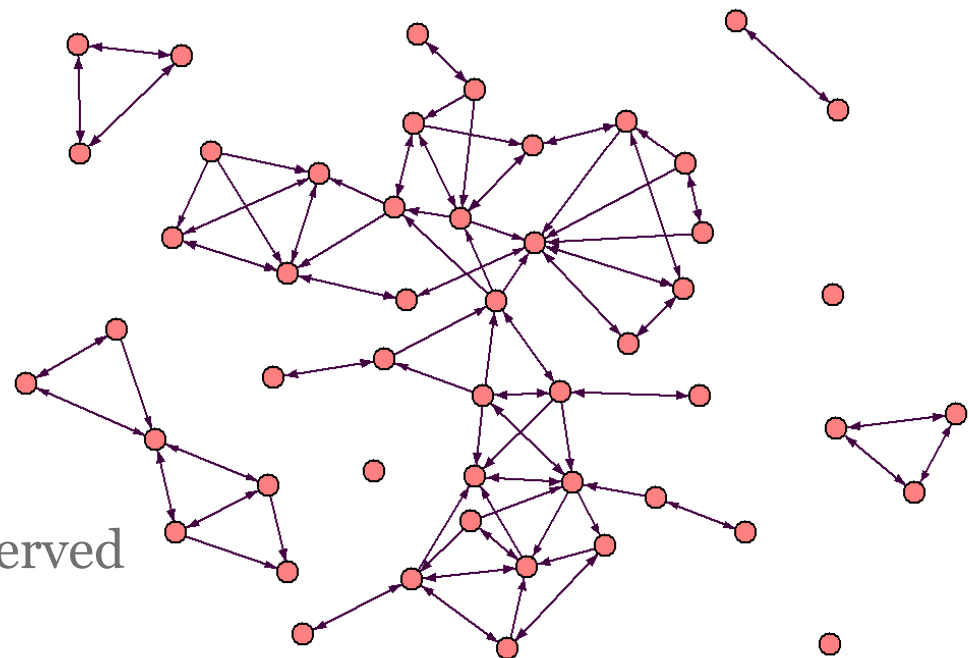
*Conclusion: by and large, the analysis supports the hypotheses...*



## Example: 50 Scottish girls' nominated friendship

› data from Michell & Amos (1997)

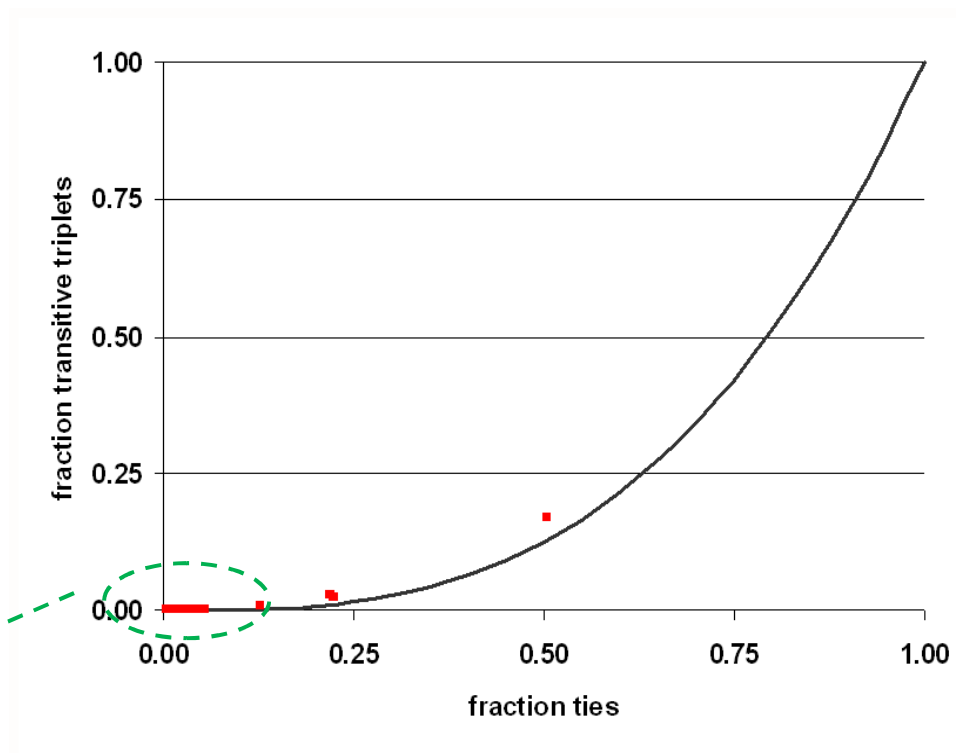
- 50 girls
- Scotland 1995
- 1<sup>st</sup> year secondary
- 13 years old



› *triangulation* can be observed



- triangulation general: typically more triangles in data than expected under random tie formation...



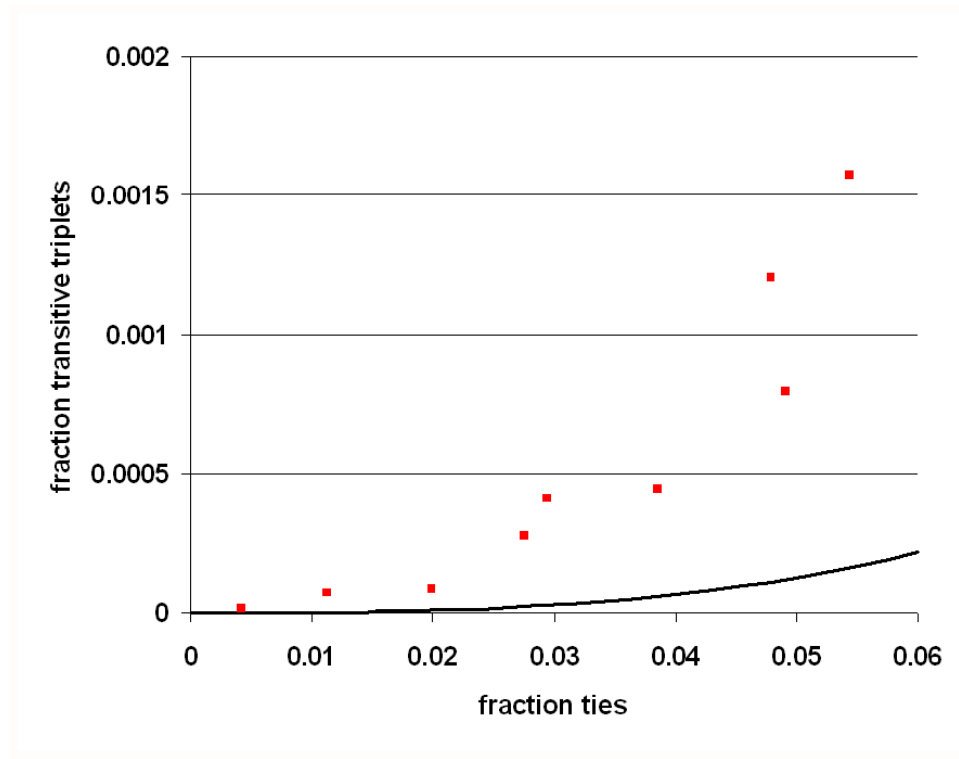
zooming in on  
 the next slide

...

Red markers  
 stand for  
 empirically  
 observed data  
 sets, black  
 line for expec-  
 tations under  
 randomness.



- › Red markers stand for observed values in several empirical data sets; the black line indicates expectations under randomness.



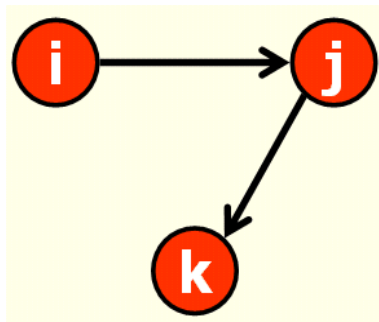
Red markers  
 lie way above  
 the random  
 expectation.

So: include  
 triangulation in  
 your model for  
 this type of  
 data!

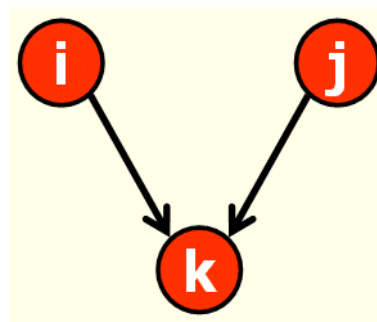


## First try: estimate a simple model with triangulation / transitive closure tendencies

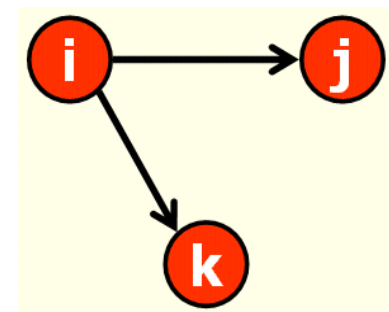
- › include *precursor configurations* in attempt to arrive at a meaningful model (Snijders 2002)



2-path



in-2-star



out-2-star



Good convergence is indicated by the t-statistics being close to zero.  
One or more of the t-statistics are rather large.  
Convergence of the algorithm is doubtful.

@2

Estimation results.

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Regular end of estimation algorithm. Total of 2092 iteration steps.

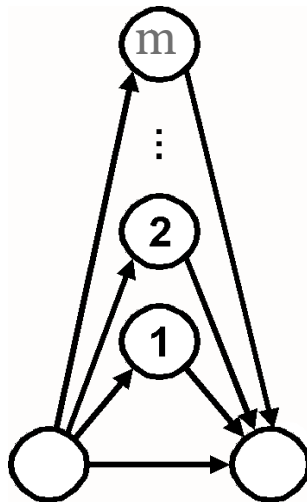
@3

Estimates and standard errors

1. u: number of arcs	4.9212	(	1.7544)
2. u: reciprocity	3.9062	(	0.3581)
3. u: in-2-stars	-1.2377	(	0.3367)
4. u: 2-paths	-0.8630	(	0.1349)
5. u: transitive triplets	-12.7060	(	99.0000)



## Second try: estimate another model with ‘new specification’ (Snijders et al. 2006)



$$s_k(x) = \sum_{m=1}^{n-2} (-1)^{m+1} \frac{\tau_m}{2^{m-1}}$$

“alternating m-triangles statistic”

$\tau_m$  = number of m-triangles  
(configurations on the left)



Good convergence is indicated by the t-statistics being close to zero.

@2

Estimation results.

-----

Regular end of estimation algorithm.  
Total of 2015 iteration steps.

@3

Estimates and standard errors

1. u: number of arcs	-3.2498	(	0.3456)
2. u: reciprocity	4.4430	(	0.3867)
3. u: alternating m-triangles, parameter 2	0.6734	(	0.0758)

**Why does estimation seem so difficult for some models?**

**Why so complex statistics like alternating m-triangles?**

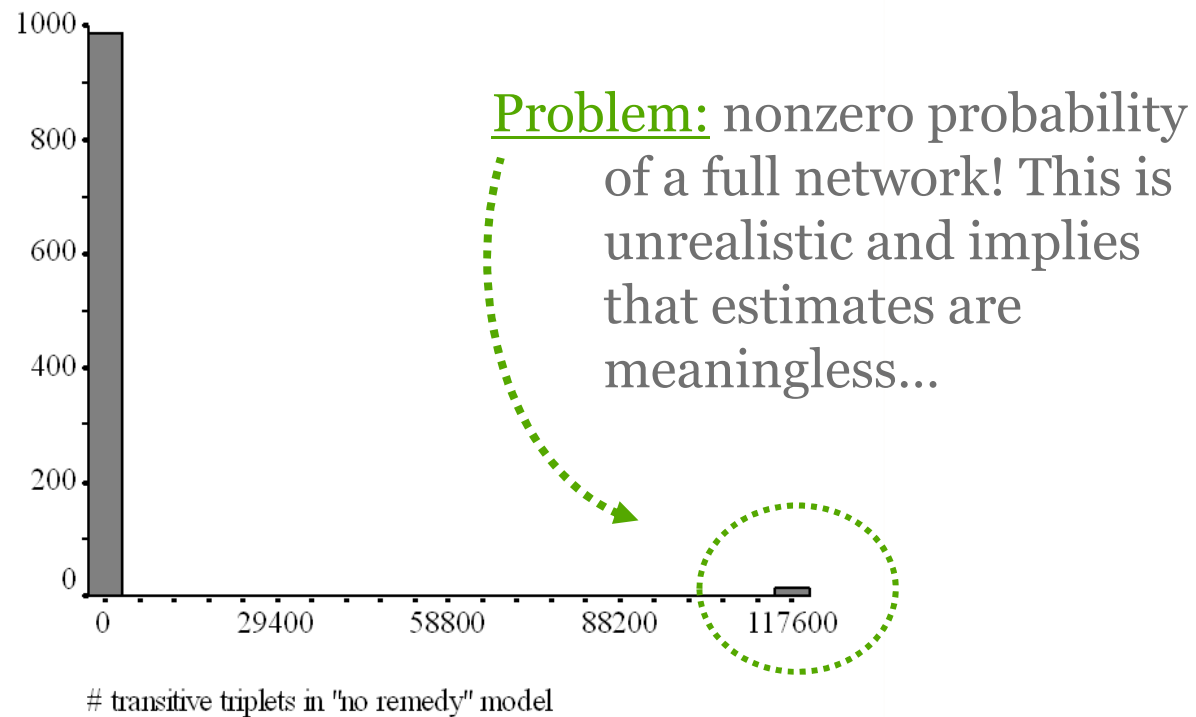


## C: Estimation of ERGMs

- › *estimation is based on simulation:*
  1. assume starting values for parameters
  2. simulate networks that are drawn from this parametrisation's ERG distribution
  3. compare them to actually observed (and to-be-modelled) network
  4. adjust parameters in order to get **better fit** [on desired dimensions]
  5. repeat from 2. until parameter values **converge**
  
- › this unfortunately does not always work, and notably triangulation effects cause problems...

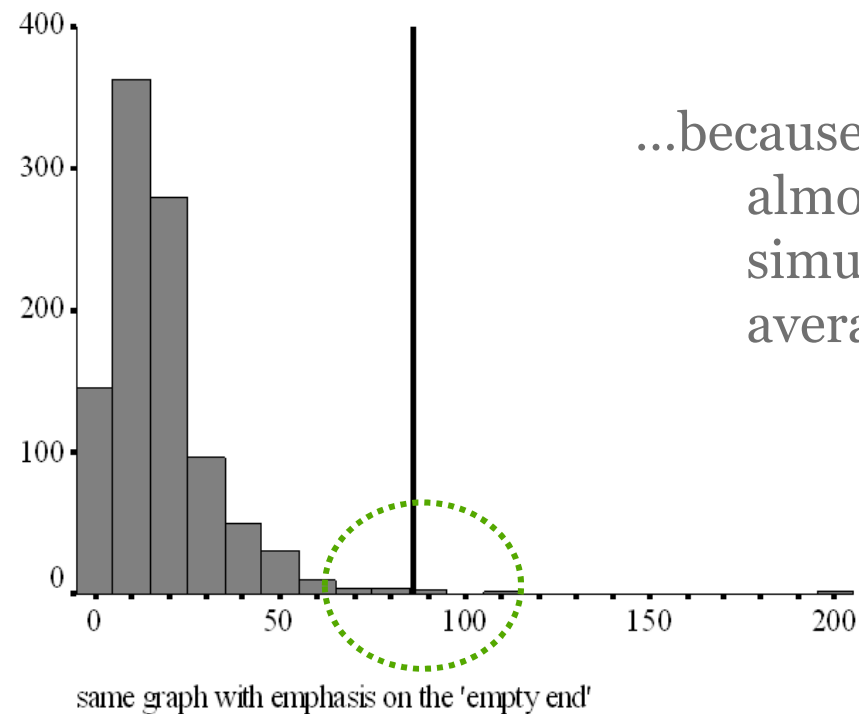


- > see here the distribution of the number of transitive triplets over simulated networks from a **model like the first** we estimated...





- › ...zooming in on the ‘low end’ (i.e., realistic networks; vertical line = observed value):



...because observed value is almost never assumed by simulations – only “on average”!



## To keep in mind about ERGMs:

- › Visualise ERG models as probability distributions on a (huge) space of all possible network,
- › one observed network is modelled as drawn from that distribution.
- › Model parameters  $\beta$  are
  - attached to network statistics  $\mathbf{s}$ ,
  - these statistics in general correspond to subgraph counts (local patterns, ‘motifs’),
  - the parameters describe the relative prevalence of the corresponding subgraph in the total graph.
- › Interpretation of parameters needs to take into account other parameters (notably edges / arcs).



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